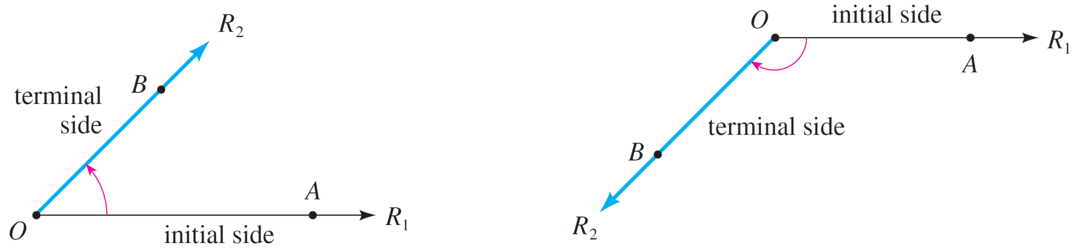
Section 6.1 Angle Measure

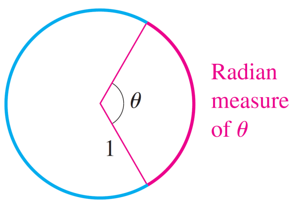
An **angle** AOB consists of two rays R_1 and R_2 with a common vertex O (see the Figures below). We often interpret an angle as a rotation of the ray R_1 onto R_2. In this case, R_1 is called the **initial side**, and R_2 is called the **terminal side** of the angle. If the rotation is counterclockwise, the angle is considered **positive**, and if the rotation is clockwise, the angle is considered **negative.**



Positive angle   Negative angle

Angle Measure

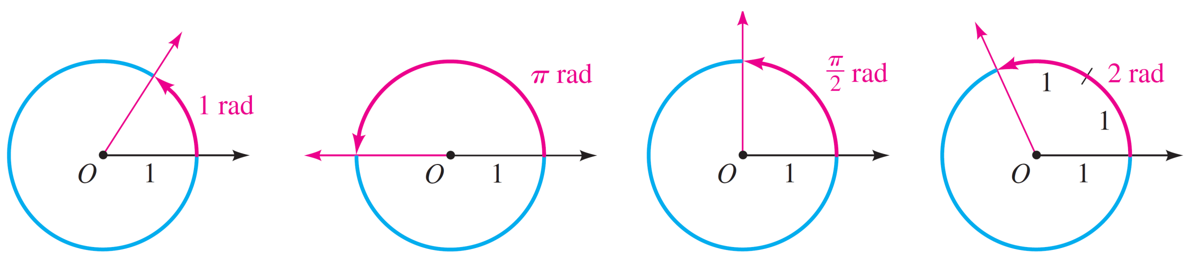
The **measure** of an angle is the amount of rotation about the vertex required to move R_1 onto R_2. Intuitively, this is how much the angle "opens." One unit of measurement for angles is the **degree**. An angle of measure 1 degree is formed by rotating the initial side $\frac{1}{360}$ of a complete revolution. In calculus and other branches of mathematics, a more natural method of measuring angles is used — *radian measure*. The amount an angle opens is measured along the arc of a circle of radius 1 with its center at the vertex of the angle.



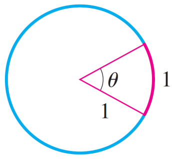
Definition of Radian Measure

If a circle of radius 1 is drawn with the vertex of an angle at its center, then the measure of this angle in **radians** (abbreviated **rad**) is the length of the arc that subtends the angle.

The circumference of the circle of radius 1 is 2π and so a complete revolution has measure 2π rad, a straight angle has measure π rad, and a right angle has measure $\pi/2$ rad. An angle that is subtended by an arc of length 2 along the unit circle has radian measure 2 (see the Figures below).



Since a complete revolution measured in degrees is $360°$ and measured in radians is 2π rad, we get the following simple relationship between these two methods of angle measurement.

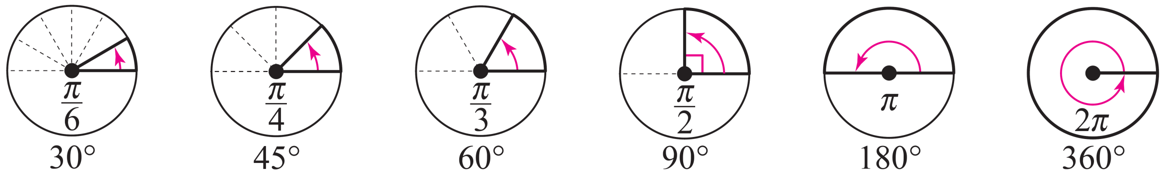


Measure of $\theta = 1$ rad
Measure of $\theta \approx 57.296°$

Relationship between Degrees and Radians

$$180° = \pi \text{ rad} \qquad 1 \text{ rad} = \left(\frac{180}{\pi}\right)^{\circ} \qquad 1° = \frac{\pi}{180} \text{ rad}$$

1. To convert degrees to radians, multiply by $\dfrac{\pi}{180}$.



To get some idea of the size of a radian, notice that

$$1 \text{ rad} \approx 57.296° \qquad \text{and} \qquad 1° \approx 0.01745 \text{ rad}$$

EXAMPLE:

(a) Express 60° in radians. (b) Express $\dfrac{\pi}{6}$ rad in degrees.

Solution:

(a) $60° = 60\left(\dfrac{\pi}{180}\right) \text{rad} = \dfrac{\pi}{3} \text{ rad}$ (b) $\dfrac{\pi}{6} \text{ rad} = \left(\dfrac{\pi}{6}\right)\left(\dfrac{180}{\pi}\right) = 30°$

EXAMPLE:

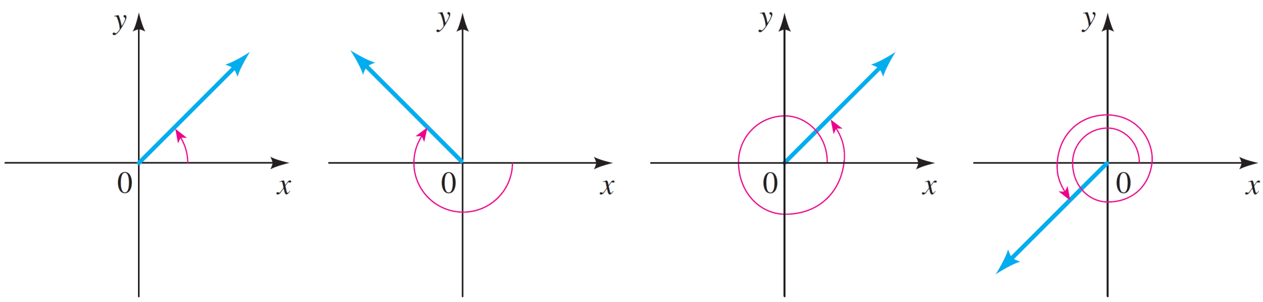
(a) Express 40° in radians. (b) Express $\dfrac{3\pi}{4}$ rad in degrees.

Solution:

(a) $40° = 40\left(\dfrac{\pi}{180}\right) \text{rad} = \dfrac{2\pi}{9} \text{ rad}$ (b) $\dfrac{3\pi}{4} \text{ rad} = \left(\dfrac{3\pi}{4}\right)\left(\dfrac{180}{\pi}\right) = 135°$

Angles in Standard Position

An angle is in **standard position** if it is drawn in the xy-plane with its vertex at the origin and its initial side on the positive x-axis. The Figures below give examples of angles in standard position.



(a) (b) (c) (d)

Two angles in standard position are **coterminal** if their sides coincide. The angles in Figures (a) and (c) above are coterminal.

EXAMPLE:

(a) Find angles that are coterminal with the angle $\theta = 30^\circ$ in standard position.

(b) Find angles that are coterminal with the angle $\theta = \dfrac{\pi}{3}$ in standard position.

Solution:

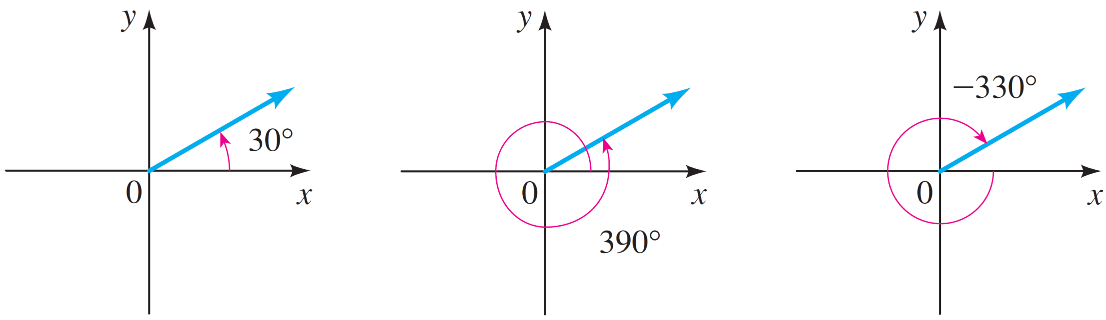
(a) To find positive angles that are coterminal with θ, we add any multiple of 360° to 30°. Thus

$$30^\circ + 360^\circ = 390^\circ, \qquad 30^\circ + 720^\circ = 750^\circ, \quad \text{etc.}$$

are coterminal with $\theta = 30^\circ$. To find negative angles that are coterminal with θ, we subtract any multiple of 360° from 30°. Thus

$$30^\circ - 360^\circ = -330^\circ, \qquad 30^\circ - 720^\circ = -690^\circ, \quad \text{etc.}$$

are coterminal with θ.



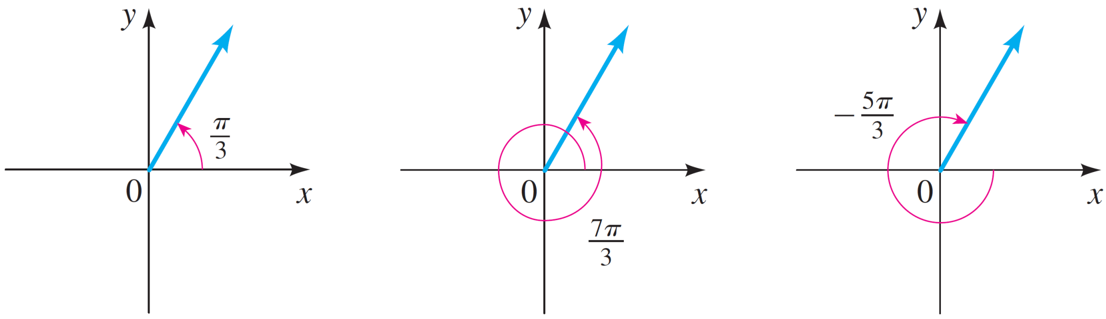
(b) To find positive angles that are coterminal with θ, we add any multiple of 2π to $\pi/3$. Thus

$$\frac{\pi}{3} + 2\pi = \frac{7\pi}{3}, \qquad \frac{\pi}{3} + 4\pi = \frac{13\pi}{3}, \quad \text{etc.}$$

are coterminal with $\theta = \pi/3$. To find negative angles that are coterminal with θ, we subtract any multiple of 2π from $\pi/3$. Thus

$$\frac{\pi}{3} - 2\pi = -\frac{5\pi}{3}, \qquad \frac{\pi}{3} - 4\pi = -\frac{11\pi}{3}, \quad \text{etc.}$$

are coterminal with θ.



EXAMPLE:

(a) Find angles that are coterminal with the angle $\theta = 62^\circ$ in standard position.

(b) Find angles that are coterminal with the angle $\theta = \dfrac{5\pi}{6}$ in standard position.

EXAMPLE:

(a) Find angles that are coterminal with the angle $\theta = 62^\circ$ in standard position.

(b) Find angles that are coterminal with the angle $\theta = \dfrac{5\pi}{6}$ in standard position.

Solution:

(a) To find positive angles that are coterminal with θ, we add any multiple of 360° to 62°. Thus

$$62^\circ + 360^\circ = 422^\circ, \qquad 62^\circ + 720^\circ = 782^\circ, \quad \text{etc.}$$

are coterminal with $\theta = 62^\circ$. To find negative angles that are coterminal with θ, we subtract any multiple of 360° from 62°. Thus

$$62^\circ - 360^\circ = -298^\circ, \qquad 62^\circ - 720^\circ = -658^\circ, \quad \text{etc.}$$

are coterminal with θ.

(b) To find positive angles that are coterminal with θ, we add any multiple of 2π to $5\pi/6$. Thus

$$\frac{5\pi}{6} + 2\pi = \frac{17\pi}{6}, \qquad \frac{5\pi}{6} + 4\pi = \frac{29\pi}{6}, \quad \text{etc.}$$

are coterminal with $\theta = 5\pi/6$. To find negative angles that are coterminal with θ, we subtract any multiple of 2π from $5\pi/6$. Thus

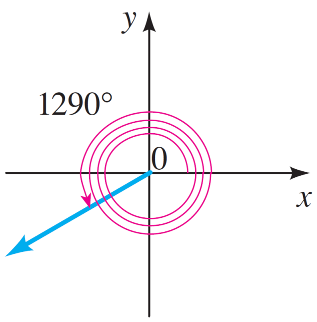
$$\frac{5\pi}{6} - 2\pi = -\frac{7\pi}{6}, \qquad \frac{5\pi}{6} - 4\pi = -\frac{19\pi}{6}, \quad \text{etc.}$$

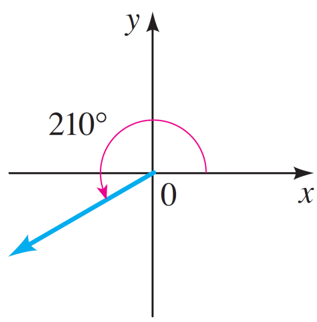
are coterminal with θ.

EXAMPLE: Find an angle with measure between 0° and 360° that is coterminal with the angle of measure 1290° in standard position.

Solution: We can subtract 360° as many times as we wish from 1290°, and the resulting angle will be coterminal with 1290°. Thus, $1290^\circ - 360^\circ = 930^\circ$ is coterminal with 1290°, and so is the angle $1290^\circ - 2(360)^\circ = 570^\circ$.

To find the angle we want between 0° and 360°, we subtract 360° from 1290° as many times as necessary. An efficient way to do this is to determine how many times 360° goes into 1290°, that is, divide 1290 by 360, and the remainder will be the angle we are looking for. We see that 360 goes into 1290 three times with a remainder of 210. Thus, 210° is the desired angle (see the Figures below).





EXAMPLE:

(a) Find an angle between 0 and 2π that is coterminal with 100.

(b) Find an angle between 0° and 360° that is coterminal with -3624°.

EXAMPLE:

(a) Find an angle between 0 and 2π that is coterminal with 100.

(b) Find an angle between 0° and 360° that is coterminal with -3624°.

Solution:

(a) We have

$$100 - 30\pi \approx 5.7522$$

(b) We have

$$-3624^\circ + 11 \cdot 360^\circ = -3624^\circ + 3960^\circ = 336^\circ$$

EXAMPLE: Find an angle between 0 and 2π that is coterminal with $\dfrac{88\pi}{3}$.

Solution: We have

$$\frac{88\pi}{3} - 28\pi = \frac{4\pi}{3}$$

EXAMPLE: Find an angle with measure between 0° and 360° that is coterminal with the angle of measure 1635° in standard position.

Solution: We have

$$1635^\circ - 4 \cdot 360^\circ = 1635^\circ + 1440^\circ = 195^\circ$$